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On Recursion Relations for Splines

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In this paper we generalize results of Müllenheim on recursion relations for splines developed for the calculation of the solution of the Hermite–Birkhoff interpolation problem and the continuous approximation of the solution of a nonlinear two-point boundary value problem. Furthermore we give simpler proofs.

1. INTRODUCTION AND RECURSION RELATIONS

Recently Müllenheim [1] developed some new recursion relations for splines which are of much use for the treatment of the Hermite-Birkhoff interpolation problem and a continuous approximation of the solution of a nonlinear two-point boundary value problem.

The object of this paper is to generalize results of Müllenheim and give simpler proofs. Let p be a spline of degree m ($\geqslant 3$) defined on a uniform partition with knots $x_i = i$ $(-\infty < i < \infty)$ and $p_i^{(\nu)} = p^{(\nu)}(i)$. Let the coefficients $c_{k,1}$ and $c_{k,0}$ be defined as in [1] by

$$c_{0,1} = 1 \qquad c_{2,1} = -\frac{1}{3!} \qquad c_{4,1} = -\frac{1}{5!} + \frac{1}{3!^2}$$

$$c_{k,1} = -\frac{c_{0,1}}{(k+1)!} - \frac{c_{2,1}}{(k-1)!} \cdots - \frac{c_{k-2,1}}{3!}$$

$$c_{k,0} = -\frac{c_{0,1}}{k!} - \frac{c_{2,1}}{(k-2)!} - \cdots - \frac{c_{k-2,1}}{2!} - c_{k,1},$$

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where

$$\sum_{\substack{k=0\\k:\text{ even}}}^{m-1} c_{k,1}/(m-k)! = 0 \qquad (m\text{: odd}). \tag{1}$$

Then for odd m, the recursion relations are as follows:

THEOREM (Müllenheim [1]). *For odd m, we have*

$$\sum_{\substack{k=0\\k:\text{ even}}}^{m-\nu} \left(c_{k,1}\, p_{i+1}^{(k+\nu-1)} + 2c_{k,0}\, p_i^{(k+\nu-1)} + c_{k,1}\, p_{i-1}^{(k+\nu-1)}\right) = 0$$

$$(\nu = 1, 3, ..., m-2). \tag{2}$$

Before we give our recursion relations, we notice that the essential equation in (2) is $\nu = 1$ since $p^{(\nu-1)}$ for odd $\nu \geqslant 3$ is considered to be a spline of degree $m-(\nu-1)$ (=odd for odd m); the other equations for odd $\nu \geqslant 3$ are easily obtained from $\nu = 1$. Therefore, we shall consider the case when $\nu = 1$. Let the coefficients $d_k(m)$ $(k = 0, 2, ..., m-1)$ be defined by

$$d_0(m) = 0, \qquad d_k(m) = \sum_{\substack{i=0\\i:\text{ even}}}^{k-2} d_{i,1}(m)/(k-i)!, \tag{3}$$

where $d_{k,1}(m)$ $(k = 0, 2, ..., m-1)$ are parameters satisfying the conditions

$$d_{0,1}(m) = 1, \qquad \sum_{\substack{k=0\\k:\text{ even}}}^{m-1} d_{k,1}(m)/(m-k)! = 0. \tag{4}$$

With the above introduced constants $d_{k,1}(m)$ and $d_k(m)$, we have

THEOREM 1. *For odd m,*

$$\sum_{\substack{k=0\\k:\text{ even}}}^{m-1} \{d_{k,1}(m)(p_{i+1}^{(k)} - 2p_i^{(k)} + p_{i-1}^{(k)}) - 2d_k(m)\, p_i^{(k)}\} = 0. \tag{5}$$

Proof. We have only to check if the above Eq. (5) is valid for $p(x) = 1$, $(x-i)$, ..., $(x-i)^m$, and $(x-i)_+^m$ since p is a linear combination of the terms on $[i-1, i+1]$.

Letting $d_{k,1}(m) = c_{k,1}$ $(k = 0, 2, ..., m-1)$ in our relation (5), we have the essential one in (2). For $m = 5$, we have a family of one-parameter relations,

$$\begin{aligned}(p_{i+1} - 2p_i + p_{i-1}) &+ \theta(p''_{i+1} - 2p''_i + p''_{i-1}) - p''_i \\ &- (1/120 + \theta/6)(p^{(4)}_{i+1} - 2p^{(4)}_i + p^{(4)}_{i-1}) - (1/12 + \theta)\, p^{(4)}_i \\ = 0 & \end{aligned} \tag{6}$$

with $\theta = d_{2,1}(5)$.

Letting $\theta = -1/20$, we have the following formula that is of much use for calculation of $p^{(4)}_i$:

$$p^{(4)}_i = 30(p_{i+1} - 2p_i + p_{i-1}) - (3/2)(p''_{i+1} + 18p''_i + p''_{i-1}). \tag{7}$$

Or letting $\theta = 0$, we have

$$p''_i = (p_{i+1} - 2p_i + p_{i-1}) - (1/120)(p^{(4)}_{i+1} + 8p^{(8)}_i + p^{(4)}_{i-1}). \tag{8}$$

Next we consider the case when m is even. Let the coefficients $\tilde{c}_{k,1}$, $\tilde{c}_{k+1,0}$, and $\tilde{c}_{k,0}$ be defined as in [1] by

$$\tilde{c}_{k,1} = -2\left(\frac{c_{k-2,1}}{4!} + \frac{c_{k-4,1}}{6!} + \cdots + \frac{c_{0,1}}{(k+2)!}\right)$$

$$\tilde{c}_{k+1,0} = -\tilde{c}_{k,1} - \frac{c_{k-2,1}}{3!} - \cdots - \frac{c_{0,1}}{(k+1)!}$$

$$\tilde{c}_{k,0} = -\tilde{c}_{k,1} - \frac{c_{k-2,1}}{2!} - \cdots - \frac{c_{0,1}}{k!} \qquad (k = 0, 2, ..., m-2).$$

Then the results of Müllenheim [1] are as follows:

THEOREM (Müllenheim [1]). *For even m, we have*

$$\begin{aligned}\sum_{\substack{k=0\\ k:\,\text{even}}}^{m-3} &(c_{k,1}\, p^{(k+\nu-1)}_{i+1} + 2c_{k,0}\, p^{(k+\nu-1)}_i + c_{k,1}\, p^{(k+\nu-1)}_{i-1}) \\ &+ (\tilde{c}_{m-\nu-1,1}\, p^{(m-2)}_{i+1} + \gamma_{m-\nu}\, p^{(m-2)}_i + \beta_{m-\nu}\, p^{(m-2)}_{i-1}) \\ &+ (\tilde{c}_{m-\nu,0}\, p^{(m-1)}_i + \alpha_{m-\nu}\, p^{(m-1)}_{i-1}) = 0 \qquad (\nu = 1, 3, ..., m-3),\end{aligned} \tag{9}$$

where

$$\alpha_\nu := \tilde{c}_{\nu,0}, \qquad \beta_\nu := \tilde{c}_{\nu-1,1} + 2\tilde{c}_{\nu,0},$$

$$\begin{aligned}\gamma_\nu := -&\sum_{\substack{k=0\\ k:\,\text{even}}}^{\nu-3} \left\{\frac{2^{\nu-k-1}}{(\nu-k-1)!}\, c_{k,1} + \frac{2}{(\nu-k-1)!}\, c_{k,0}\right\} \\ &-2\tilde{c}_{\nu-1,1} - 2\tilde{c}_{\nu,0} \qquad (\nu = 3, 5, ..., m-1).\end{aligned}$$

For even m, let the coefficients $d_k(m)$ $(k=0, 2, ..., m)$ be defined by (3), where $d_{0,1}(m)=1$ but we do not require the second condition of (4). Similarly as in the case when m is odd, note that one essential equation in (9) is $\nu=1$. For $\nu=1$, as in the proof of Theorem 1 we have

THEOREM 2. *For even m,*

$$\sum_{\substack{k=0\\ k:\text{even}}}^{m-2} \{d_{k,1}(m)(p_{i+1}^{(k)} - 2p_i^{(k)} + p_{i-1}^{(k)}) - 2d_k(m)\, p_i^{(k)}\}$$

$$- d_m(m)(p_{i+1}^{(m-1)} - p_{i-1}^{(m-1)}) = 0, \tag{10}$$

where $p_{i+1}^{(m-1)} - p_{i-1}^{(m-1)} = 2(p_{i+1}^{(m-2)} - 2p_i^{(m-2)} + p_{i-1}^{(m-2)})$, *since* $p^{(m-2)}$ *is quadratic on* $[i-1, i]$ *and* $[i, i+1]$.

Letting $d_{k,1}(m) = c_{k,1}$ $(k=0, 2, ..., m-2)$, then we easily have (i) and (ii) except (iii):

$$\begin{aligned}
&\text{(i)} \quad -2\{d_{k,1}(m) + d_k(m)\} = 2c_{k,0} \qquad (k=0, 2, ..., m-4)\\
&\text{(ii)} \quad d_{m-2,1}(m) - 2d_m(m) = \tilde{c}_{m-2,1} \qquad (\text{or } = \beta_{m-1} - 2\alpha_{m-1})\\
&\text{(iii)} \quad 2\{2d_m(m) - d_{m-2}(m) - d_{m-2,1}(m)\} = \gamma_{m-1} + 2\alpha_{m-1}.
\end{aligned} \tag{11}$$

Here we shall prove only (iii). From (ii) and $\alpha_{m-1} = \tilde{c}_{m-1,0}$, Eq. (iii) is equivalent to

$$d_{m-2}(m) = \sum_{\substack{k=0\\ k:\text{even}}}^{m-4} \left\{\frac{2^{m-k-3} c_{k,1}}{(m-k-2)!} + \frac{c_{k,0}}{(m-k-2)!}\right\}$$

$$+ \sum_{\substack{k=0\\ k:\text{even}}}^{m-4} \left\{\frac{2^{m-k-3} d_{k,1}(m)}{(m-k-2)!} - \frac{d_{k,1}(m) + d_k(m)}{(m-k-2)!}\right\}$$

or

$$d_{m-2}(m) + \sum_{\substack{k=0\\ k:\text{even}}}^{m-4} \frac{d_k(m)}{(m-k-2)!} = \sum_{\substack{k=0\\ k:\text{even}}}^{m-4} \frac{2^{m-k-3} - 1}{(m-k-2)!} d_{k,1}(m). \tag{12}$$

This identity can easily be obtained by comparing the coefficients of $d_{k,1}(m)$ $(k=0, 2, ..., m-4)$ on both sides of (12) where the following relation is of use:

$$\binom{k}{0} + \binom{k}{2} + \cdots + \binom{k}{k-2} = 2^{k-1} - 1 \qquad (k\text{: even}).$$

Hence, by means of (11) our relation (10) becomes

$$
\begin{aligned}
0 = & \sum_{\substack{k=0 \\ k:\,\text{even}}}^{m-4} (c_{k,1} p_{i+1}^{(k)} + 2c_{k,0} p_i^{(k)} + c_{k,1} p_{i-1}^{(k)}) + \{d_{m-2,1}(m) - 2d_m(m)\}\, p_{i+1}^{(m-2)} \\
& + 2\{2d_m(m) - d_{m-2}(m) - d_{m-2,1}(m)\}\, p_i^{(m-2)} \\
& + \{d_{m-2,1}(m) - 2d_m(m)\}\, p_{i-1}^{(m-2)} \\
= & \sum (\cdots) + \tilde{c}_{m-2,1}\, p_{i+1}^{(m-2)} + (\gamma_{m-1} + 2\alpha_{m-1})\, p_i^{(m-2)} \\
& + (\beta_{m-1} - 2\alpha_{m-1})\, p_{i-1}^{(m-2)} \\
= & \sum (\cdots) + (\tilde{c}_{m-2,1}\, p_{i+1}^{(m-2)} + \gamma_{m-1}\, p_i^{(m-2)} + \beta_m\, p_{i-1}^{(m-2)}) \\
& + (\alpha_{m-1}\, p_i^{(m-1)} + \tilde{c}_{m-1,0}\, p_{i-1}^{(m-1)}),
\end{aligned} \tag{13}
$$

where $\alpha_{m-1} = \tilde{c}_{m-1,0}$. Thus we obtain the essential equation in (9) as a special case of our recursion relation (10).

For $m = 4$, we have a family of one-parameter relations,

$$
\begin{aligned}
& (p_{i+1} - 2p_i + p_{i-1}) + \theta(p''_{i+1} - 2p''_i + p''_{i-1}) - p''_i \\
& \quad - (1/24 + \theta/2)(p_{i+1}^{(3)} - p_{i-1}^{(3)}) = 0,
\end{aligned} \tag{14}
$$

with $\theta = d_{2,1}(4)$.

Letting $\theta = -1/12$, we have the well-known short term recurrence relation for a quartic spline,

$$
(p_{i+1} - 2p_i + p_{i-1}) - (1/12)(p''_{i+1} + 10p''_i + p''_{i-1}) = 0. \tag{15}
$$

For $m = 6$, we have a family of two-parameter relations,

$$
\begin{aligned}
& (p_{i+1} - 2p_i + p_{i-1}) + \theta(p''_{i+1} - 2p''_i + p''_{i-1}) - p''_i \\
& \quad + \gamma(p_{i+1}^{(4)} - 2p_i^{(4)} + p_{i-1}^{(4)}) - (1/12 + \theta)\, p_i^{(4)} \\
& \quad - (1/720 + \theta/24 + \gamma/2)(p_{i-1}^{(5)} - p_{i-1}^{(5)}) = 0
\end{aligned} \tag{16}
$$

with $\theta = d_{2,1}(6)$ and $\gamma = d_{4,1}(6)$.

Letting $(\theta, \gamma) = (-1/30, 0)$ or $(0, -1/360)$, we have

$$
p_i^{(4)} = 20(p_{i+1} - 2p_i + p_{i-1}) - (2/3)(p''_{i+1} + 28p''_i + p''_{i-1}) \tag{17}
$$

(for this formula, see [2, p. 157]) or

$$
p''_i = (p_{i+1} - 2p_i + p_{i-1}) - (1/360)(p_{i+1}^{(4)} + 28p_i^{(4)} + p_{i-1}^{(4)}). \tag{18}
$$

In Section 2, we shall give other recursion relations which are of much use for calculation of p_i' as a linear combination of p_i and p_i'' specified in Hermite–Birkhoff interpolation problem.

2. Other Recursion Relations

Let θ and γ be any real constants. For odd m, let the coefficients $c_k(m)$ $(k=0, 2, ..., m-1)$ be defined by

$$c_0(m) = \theta + \gamma/2$$
$$\sum_{\substack{i=0 \\ i:\text{even}}}^{k} c_i(m)/(k+1-i)! = \theta/k! \qquad (k=2, ..., m-1). \tag{19}$$

As in the proof of Theorem 1, we have Theorem 3. For odd m, the following relation holds:

$$\sum_{\substack{k=0 \\ k:\text{even}}}^{m-1} c_k(m)(p_{i+1}^{(k)} - p_{i-1}^{(k)}) = \theta p_{i-1}' + \gamma p_i' + \theta p_{i+1}'. \tag{20}$$

For $m=5$, letting $(\theta, \gamma) = (7, 16)$ or $(0, 1)$ gives

$$(7p_{i+1}' + 16p_i' + 7p_{i-1}') = 15(p_{i+1} - p_{i-1}) + (p_{i+1}'' - p_{i-1}'') \tag{21}$$

or

$$p_i' = (1/2)(p_{i+1} - p_{i-1}) - (1/12)(p_{i+1}'' - p_{i-1}'') + (7/720)(p_{i+1}^{(4)} - p_{i-1}^{(4)}). \tag{22}$$

By making use of (21) or (7) – (22), we can easily construct the solution of the Hermite–Birkhoff interpolation problem, where given real data y_i, M_i, a spline p of degree m $(=5)$ with simple knots $x_i = i$ is looked for such that

$$p_i = y_i, \qquad p_i'' = M_i.$$

For even m, let the coefficients $c_k(m)$ $(k=0, 2, ..., m-4)$ be defined by (19), where k runs from 2 to $m-4$ (step 2). Next let $c_{m-2}(m)$ and $c_m(m)$ be determined by

$$c_{m-2}(m) + c_m(m) = \theta/(m-2)! - \sum_{\substack{i=0 \\ i:\text{even}}}^{m-4} c_i(m)/(m-1-i)!$$

$$c_{m-2}(m)/2 + c_m(m) = \theta/(m-1)! - \sum_{\substack{i=0 \\ i:\text{even}}}^{m-4} c_i(m)/(m-i)!.$$

Then, similarly as in the proof of Theorem 1 we have Theorem 4. For even m, the following relation holds:

$$\sum_{\substack{k=0\\k:\,\text{even}}}^{m-2} c_k(m)(p_{i+1}^{(k)} - p_{i-1}^{(k)}) + c_m(m)(p_{i+1}^{(m-1)} + p_{i-1}^{(m-1)}) = \theta p'_{i+1} + \gamma p'_i + \theta p'_{i-1}. \tag{23}$$

For $m = 6$, letting $(\theta, \gamma) = (1, 2)$ in (23) gives

$$(p'_{i+1} + 2p'_i + p'_{i-1}) = 2(p_{i+1} - p_{i-1}) + (1/6)(p''_{i+1} - p''_{i-1}) - (1/360)(p_{i+1}^{(4)} - p_{i-1}^{(4)}). \tag{24}$$

Here, an alternating sum obtained by writing down Eq. (24), subtracting (24) with i replaced with $i+1$, adding (24) with i replaced with $i+2$, and so on is equal to the short term recursion relation at two adjacent knots $x = i$ and $i+1$:

$$(p'_{i+1} + p'_i) = 2(p_{i+1} - p_i) + (1/6)(p''_{i+1} - p''_i) - (1/360)(p_{i+1}^{(4)} - p_i^{(4)}). \tag{25}$$

Recursion relations (17) and (25) would be of use for calculation of p'_i, i.e., the continuous solution of the Hermite–Birkhoff interpolation problem.

References

1. G. Müllenheim, Some new recursion relations for spline functions with application to spline collocation, *J. Approx. Theory* **58** (1989), 202–212.
2. R. Usmani, Spline solutions for nonlinear two point boundary value problems, *Int. J. Math. Sci.* **3** (1980), 151–167.